**Azem HYSA**[1]
**Marsida KLEMO**[2]

# INVESTIGATION OF MANIFOLDS IN THE CR3BP

## Abstract

Many of the problems facing physicists and applied mathematicians involve difficulties as nonlinear governing equations, variable coefficients, and nonlinear boundary conditions at complex known or unknown boundaries that preclude solving them exactly. Manifolds and optimal control were used to better understand trajectories in the circular restricted three-body problem. Equations of this problem were used to generate two-dimensional and three-dimensional stable and unstable invariant manifolds.

The instability of periodic orbits and similar periodic solutions can be exploited to analyze paths to and from every point on a given orbit. This work presents a systematic method for the designin of impulsive low-energy transferes between the Earth and the Moon by the explicit using invariant manifold theory. Invariant manifolds are tube-like structures along which a spacecraft may travel using no energy and this techniques usually only provied trajectories for uncontrolled spacecraft.

The numerical integration requires an initial position and velocity so that it can generate a trajectory over a specified time interval. A zero initial velocity is required to find the stable manifold to travel to a Lagrange point. The stable manifold can be propagated forward and backward in time so that a spacecraft is able to travel to and from a Lagrange point on the manifold. Finally, this paper discusses the tools used to generate the trajectories in this work. In this study all units of physical quantities are non-dimensional form and the results are plotted using numerical methods in MATLAB.

**Keywords:** Invariant manifold, nondimensional, low-energy transfer trajectories.

[1] Applied and Natural Sciences Department, "Aleksander Moisiu" University, Durres, Albania

[2] Applied and Natural Sciences Department, "Aleksander Moisiu" University, Durres, Albania

## 1. Introduction

Humans have not traveled to the moon since 1972, and that is the farthest we have travelled. If this mission develops, it will be the farthest humans have traveled in the solar system. This would encourage practical technology development, for creating the longest manned mission would introduce many constraints, but also could provide many answers. This would be a critical stepping stone in developing technology for manned missions to other planets, such as Mars. It could also provide valuable scientific information about the far side of the moon, and humans could deploy instruments or control a rover.

In 2000, Koon et al., constructed a planar lunar transfer that was almost entirely ballistic using the techniques involved in Conley's method. Following (Conley, Koon et al. 2000) observed that the planar libration orbits act as a gateway between the interior and exterior regions of space about the constructed a trajectory that targets the interior of the stable invariant manifold of a planar libration orbit about the Earth–Moon $L_2$ point. Once inside the interior of the stable manifold, the spacecraft ballistically arrives at a temporarily captured orbit about the Moon. Many authors have debated what it means to be temporarily captured at the Moon; Koon et al., define a similar term, "ballistically captured" to be a trajectory that comes within the sphere of influence of the Moon and revolves about the Moon at least once. There is much literature on invariant manifolds and connecting orbits in the CR3BP; see for example G´omez et al. (2004); Koon et al. (2008); Lo and Ross (1997); Davis et al. (2010, 2011); Tantardini et al. (2010).

Further advances have been made since 2004 to apply dynamical systems theory to the generation of three-dimensional low-energy lunar transfers. Parker mapped out numerous families of low-energy transfers, illuminating different geometries that are available for spacecraft to travel to the Moon and arrive in lunar libration orbits without requiring any capture maneuver. Several authors have begun applying low-thrust techniques to further improve low-energy transfers, including transfers from the Earth to the Moon and transfers from one libration orbit to another. In 60 years, research has advanced the knowledge of lunar transfers from the early spacecraft missions that implemented direct lunar transfers to modern analyses that reveal maps of entire families of low-energy transfers to the Moon. The Earth-Moon Lagrange point on the back side of the moon, commonly known as $L_2$, has been of recent interest in the space community, specifically for a human mission. The $L_2$ point is a point of neural gravity from the gravity fields of the Earth and the Moon and located on the far side of the moon. A human mission to the Earth-Moon Lagrange point, $L_2$, has recently been a point of interest. This topic was discussed most recently at the International Astronautical Congress (IAC) in Naples in October 2012 by space agencies such as NASA and Boeing.

## 2. Methodology and Invariant Manifolds

This section introduces the methodology used in the analysis and construction of invariant manifolds. The paper begins by simply defining the physical constants used in these analyses, including the masses and radii of the Earth, the Moon, and the small particle in the Earth-Moon system. It then defines the time systems used, coordinate frames, and models, including the circular restricted three-body problem. This problem describes a dynamical model that is used to characterize the motion of a small particle, in the presence of two massive bodies. The model assumes the two massive bodies orbit their barycenter in circular orbits. Coordinate systems include a reference frame and an origin, and are often rotating or translating relative to other bodies. A coordinate system is inertial only when it is not accelerating. When referencing motion in the Solar System, the only truly "inertial" coordinate system is one that is not rotating and centered at the Solar System barycenter. Strictly speaking, no Earth-centered coordinate system can be inertial, even one that is not rotating, since the Earth is accelerating in its orbit as it revolves about the Sun. Although it is inaccurate, coordinate systems may be referred to in this paper as "inertial" when they are merely nonrotating.

In order to generate a manifold for this problem, the two or three-dimensional nonlinear equations of motion were numerically integrated. The state-space representation of the equation of motion was programmed into MATLAB. The set of four or six, first order, nonlinear equations were numerically integrated using the Runge-Kutta method.

Especially, all asymptotic orbits, which are asymptotic to the periodic orbit, form a tube which is called invariant manifold, can present lots of advantageous of mission design. Invariant manifold is also a boundary that separates the transit and non-transit tubes. Transit orbits are always in invariant manifold tube and can pass to one region to another. Invariant manifold are depending on periodic orbits around equilibrium points, and they can be computed thanks to them. But there are various periodic and quasi periodic around there, so which Asymptotic orbits relate which periodic orbits? This is very easy, while in Planar-CR3BP there is only one unique periodic orbit around unstable equilibrium point in each specific energy. So that Asymptotic orbits and so invariant manifold must be related with this periodic orbit. But in three-dimensional space, there are more than one periodic orbits in each specific energy, and all these periodic and quasi-periodic orbits have their own asymptotic orbits and so invariant manifold. But in practically, three-dimensional-invariant manifold is computed with using either three-dimensional Periodic orbits, for easiness. In addition, a transit orbit seems out side of the manifold tubes while in three-dimensional space, this means that this transit orbits actually in another invariant manifold of another periodic orbits.

These orbits are important for equilibrium point mission and capture transfer missions. They provide a way to get periodic to planet or moon, also capture trajectories are always inside the manifold tubes, and their intersection on Poincare surface of section provides optimum jumping point from one manifold to another. These orbits are used for 'Low Energy Transfer'; they can be reduced mission cost, and have great potential for any flexible and rescue missions. Same as periodic orbits, there are two main way to compute these orbits, numeric method and LP method for analytic approach. Other efficient methods for computing invariant manifolds include semi-analytical approximations (Jorba et al. 1999; Alessi et al. 2009; G´omez and Mondelo 2001). The latter methods are very precise in a neighborhood of the center of expansion, and rely on other methods to extend the manifolds outside these neighborhoods (G´omez et al. 2001). Invariant manifold techniques around libration points have been used successfully in mission design (Lo et al. 2004). The Genesis spacecraft mission, designed to collect samples of solar wind and return them to the Earth (Lo et al. 2001), is often considered as the first mission to use invariant manifolds for its planning, while other missions have used libration point techniques (Dunham and Farquhar 2003). Having a precise idea of the geometry of invariant manifolds and their connections is desirable in the design of complex low thrust missions.

## 3. Results

To generate trajectories that meet a variety of mission constraints, tools that provide insight into the available solution space are essential. The Poincare map is a powerful tool that, in combination with a constraint on the energy level, allows a reduction in dimension such that, for the planar problem, the system is reduced into two-dimensions and the phase space is fully represented by the projection onto a plane.

We compute the intersection of the unstable manifold $M_{L1}^{U}$ from $L_1$ and the stable manifold $M_{L2}^{S}$ from $L_2$ with the space $\Sigma_1$. Of course, we can do the symmetric counterpart: stable manifold $M_{L1}^{S}$ from $L_1$ and unstable manifold $M_{L2}^{U}$ from $L_2$ with the space $\Sigma_2$ (see Figuure 1, A.Hysa, M. Klemo 2017).

Figure 1 shows 4 manifolds in the "neck" region in the Earth-Moon system, two periodic orbits around fixed points and the location of the Poincare sections. Unstable $M_{L1}^{U}$ and stable $M_{L2}^{S}$ manifolds respectively from $L_1$ and $L_2$ stopping at the plane $\Sigma_1$. Stable $M_{L1}^{S}$ and unstable $M_{L2}^{U}$ manifolds stopping at the plane $\Sigma_2$.

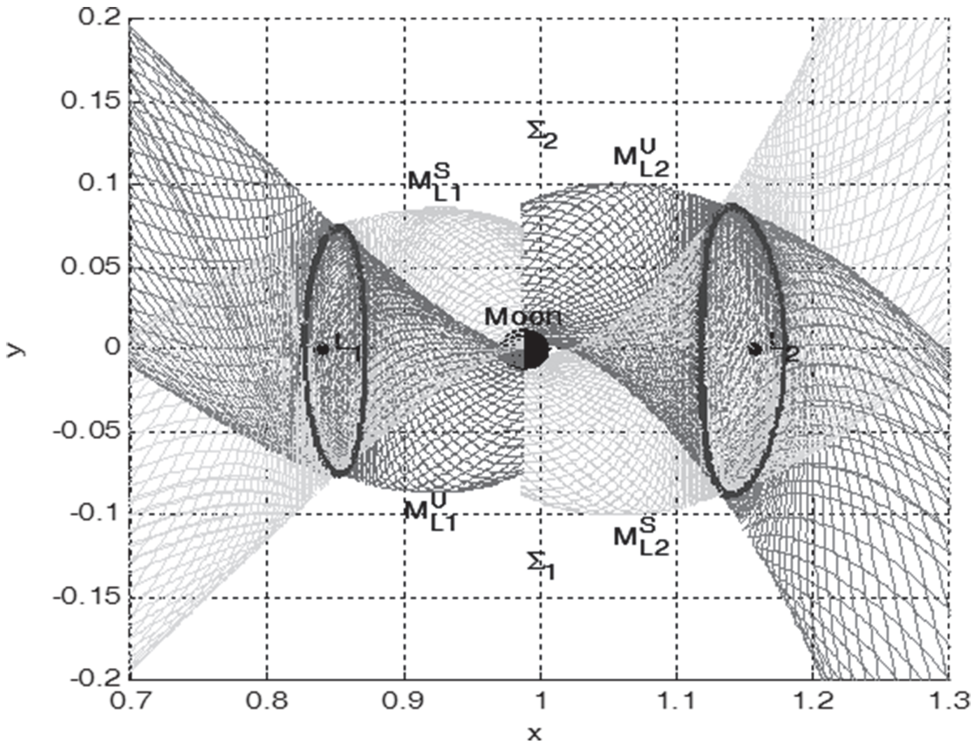


Fig.1. Stable (green) and unstable (red) manifolds associated with $L_1$ and $L_2$ periodic orbits (blue), respectively. The location of the Poincare sections $\Sigma_1$ (yellow) and $\Sigma_2$ (purple) are also shown.

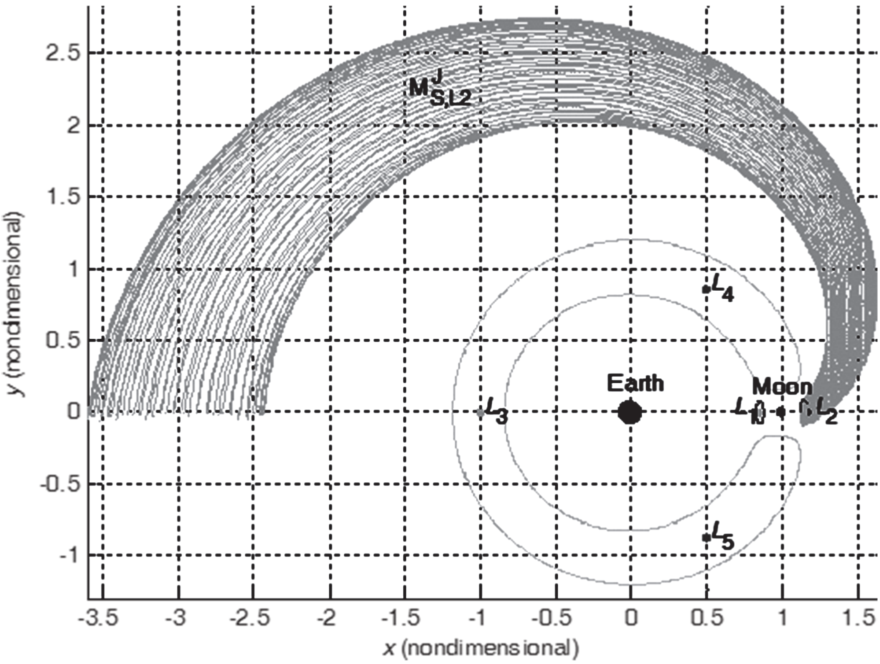


Figure 2. The stable invariant manifolds for an orbit about $L_2$ point of the rotating Earth-Moon system.

The stable and unstable manifolds for a symmetric orbit are themselves symmetric about the *x* axis. This is because the two experience equal and opposite rotation rates: counterclockwise for forward time (unstable) and clockwise for backward time (stable). Note that only one manifold exists for a certain direction of time. In other words, stable perturbations while moving forward in time do not create a stable manifold. They will simply damp out quickly, leaving no change. The same is true of unstable perturbations backward in time. If a particle is randomly perturbed while on the orbit, the particle will fall off of the orbit at an exponential rate. In Hamiltonian systems, not only are there asymptotic orbits departing from the unstable orbits, there are also asymptotic orbits approaching the unstable orbits.

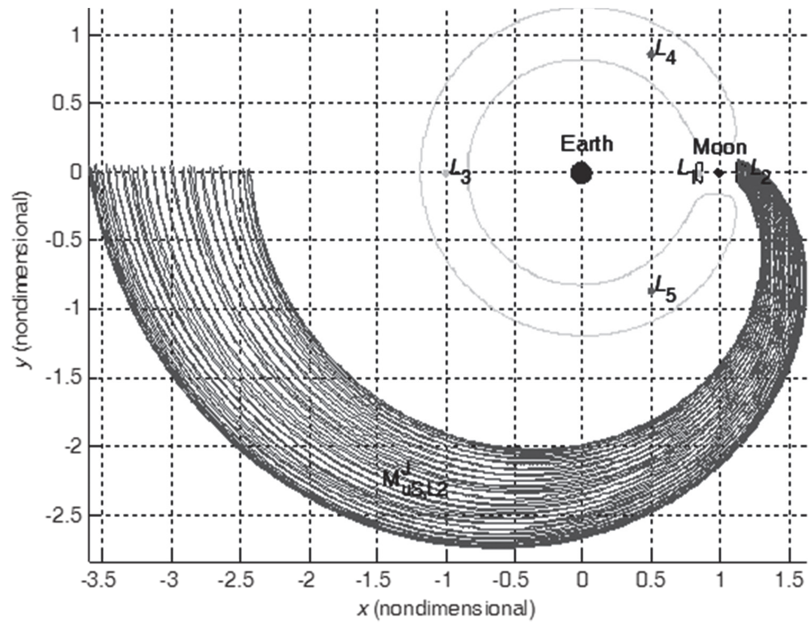


Figure 3. The unstable invariant manifolds for an orbit about $L_2$ point of the rotating Earth-Moon system.

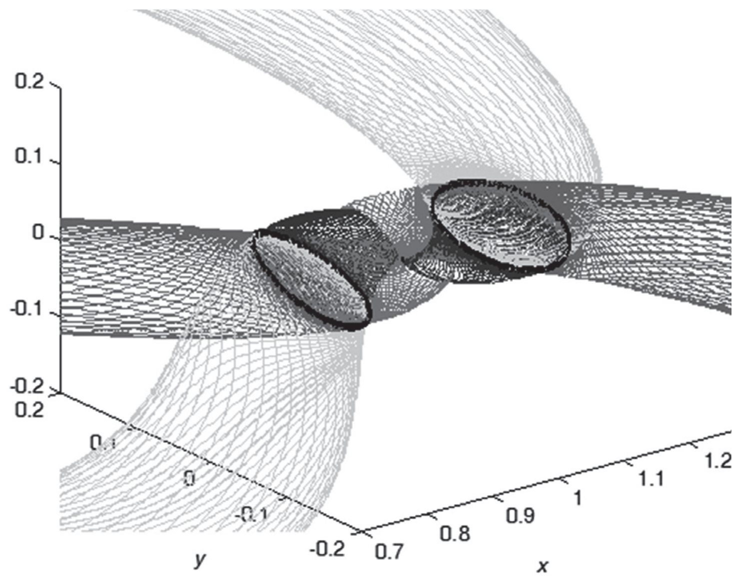


Figure 4. Three-dimensional invariant manifold of $L_1$ and $L_2$. Green trajectories are stable manifold of outside, red trajectories are unstable manifold of outside, blue trajectories are stable manifold of inside, and purple trajectories are unstable manifold of inside.

The set of all trajectories that asymptotically depart from the unstable orbit is known as the orbit's unstable invariant manifold; the set of all trajectories that asymptotically arrive onto the unstable orbit is likewise known as the orbit's stable invariant manifold. Figure 2 show the stable invariant manifold for a typical periodic orbit about the Earth-Moon $L_2$ point and Figure 3 show the unstable invariant manifold. The manifolds are very similar for orbits about the $L_1$ point. One can see the underlying tubular structure in the manifolds. As the manifolds approach one of the primaries, this structure begins to break down due to the large divergent behavior near the primaries.

Mission designers may use these invariant manifolds to model the motion of spacecraft in their vicinity. If a mission's objective is to transfer onto an unstable periodic orbit, then the spacecraft need only target that orbit's stable manifold in order to insert into that orbit. Although missions such as ISEE3 and Hiten were not designed using invariant manifolds explicitly, the underlying dynamics may be understood using invariant manifold theory. The advantage of the dynamical systems approach is the ability to compute and visualize global families of low-energy transfer trajectories, giving mission designers a priori knowledge of the underlying dynamics in the libration orbit regime.

In Figure 4, hot color orbits (red and pink) are unstable manifold of equilibrium points which are departure from equilibrium point, and cool color orbits (green and blue) are stable orbits of equilibrium points which are arrival to equilibrium point.

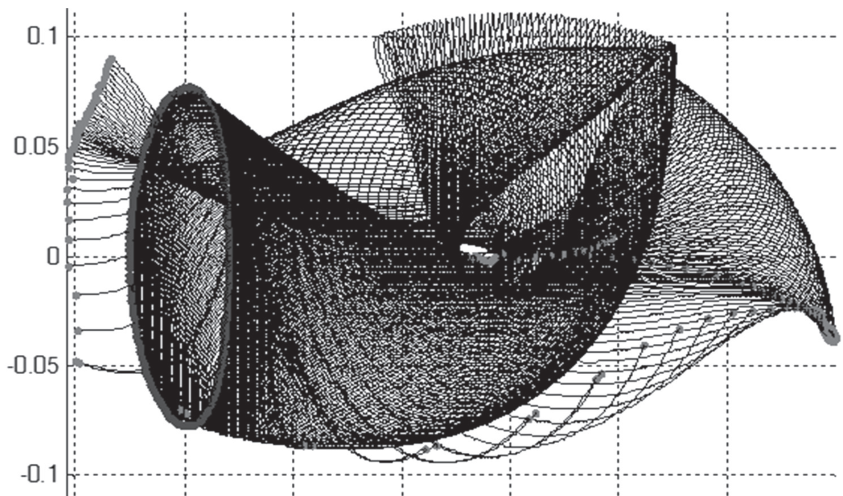


Figure 5. Unstable invariant manifold of $L_1$

In Figure 5, a wild unstable invariant manifold of $L_1$ is shown, which might be actually any invariant manifold of any equilibrium point. As explained in methods and invariant manifolds section, an invariant manifold is obtained by disturbing a periodic orbit around $L_1$: the initial conditions of each trajectory that forms the tube is obtained by making a tiny disturbance to a periodic orbit, at a particular phase. So the periodic orbit is subdivided into a series of discrete points and each point is denoted by a trajectory number which is also called "manifold number" in this paper for easiness. This will compose one strand of the invariant manifold.

## 4. Conclusions

The instability of periodic orbits and similar periodic solutions can be exploited to analyze paths to and from every point on a given orbit. While there are infinitely many of these paths, they all belong to a well-defined set called a manifold. Figure 2 shows two manifolds in the Earth-Moon system. The first, in green, is the stable manifold. This is the set of trajectories moving forward in time that asymptotically approach the periodic orbit. In contrast, the second plot in red depicts the unstable manifold, which departs the periodic orbit over time.

This paper has demonstrated how invariant manifold theory may be used to construct and understand three-dimensional ballistic lunar transfers. A spacecraft on such a transfer could remain on the periodic orbit, freely transfer to another libration orbit, freely transfer to a temporarily captured orbit about the Moon, or perform another maneuver to inject into any lunar orbit. The transfer implements three-dimensional libration orbits as staging orbits in the Earth-Moon three-body systems. Other orbits could be used to produce similar results. The advantages of the three-dimensional approach, compared with the two-dimensional work produced by Koon et al., include access to inclined lunar orbits, the option to use inclined Low-Earth-Orbit parking orbits, access to other regions of space in the Earth's neighborhood, and better communication geometry. The Shoot the Moon transfer requires less energy than conventional Hofmann transfers, but requires a much longer transfer time. Such low-energy transfers are useful for cargo transport and robotic missions to lunar orbits or to the surface of the Moon.

The approach used in this paper is robust enough to identify families of ballistic lunar transfers, including transfers that use other types of unstable orbits as staging orbits. These families will be explored and presented in future papers. Further analysis of the structure of the invariant manifolds of three-dimensional orbits in the spatial CR3BP may provide additional understanding of these lunar transfers, which may present alternative approaches for the construction and analysis of ballistic lunar transfers. The methods presented by Gómez et al. 2004, for example, may offer such an alternative.

## 5. Acknowledgements

The authors are thankful to the reviewers for their valuable suggestions to enhance the quality of our article.

## REFERENCES


1. Jeffrey S. Parker and Martin W. Lo, "Shoot the Moon 3d", AAS 05-383, 2002.
2. G. Gomez, W S Koon, M W. Lo, J E. Marsden, J Masdemont and S D. Ross, "Connecting orbits and invariant manifolds in the spatial restricted three-body problem", Institute of Physics Publishing, Nonlinearity 17 (2004) 1571–1606, PII: S0951-7715(04)67794-2.
3. Raquel L. Jarabek, M.S, Investigation of manifolds and optimized trajectories in the three-body-problem, 2004.
4. R. C. Calleja, E. J. Doedel, A. R. Humphries, A. Lemus-Rodr´ıguez, B. E. Oldeman, "Boundary-Value Problem Formulations for Computing Invariant Manifolds and Connecting Orbits in the Circular Restricted Three Body Problem", arXiv:1111.0032v2 [math.DS] 17 May 2012.
5. Morring Jr, F, Svitak, A. (2012, Oct 8). Next Stop Station-derived base at Earth-Moon L-2 is attractive, but details need work. Aviation Week & Space Technology*,* 26-27.
6. Nick Truesdale, "Using Invariant Manifolds of the Sun-Earth L2 Point for Asteroid Mining Operations", ASEN 5050 Spaceight Dynamics 12/19/2012.
7. Ann dietrich, "Visibility from Earth-Moon L2 orbits to the Moon and Earth" asen 5050 final project 12/13/2012.
8. Asli. U, "Low cost capture using multi-body dynamics", 2013. Surrey Space Center 2013.Guildford, Surrey, GU2 7XH, U.K.
9. Amanda F. Haapala, Kathleen C. Howell, "Representations of higher-dimensional Poincaré maps with applications to spacecraft trajectory design", Acta Astronautica 96 (2014) 23–41.
10. Maxime. Ch, Thomas. H and Emmanuel. T, "Low-thrust lyapunov to lyapunov and halo to halo missions with $L_2$ minimization, arXiv:1511.02089v2 [math.OC] 1 Jul 2016.
11. A. Safiya Beevi and Ram Krishan Sharma, "Effect of Oblateness of the More Massive Primary on Periodic Orbits in the Restricted Three-Body Problem", Advances in Astrophysics, Vol. 2, No. 1, February 2017.
12. A. Hysa, M. Klemo, "Invariant manifolds and space mission design", 60th International Conference for Students of Physics and Natural Sciences, Open Readings 2017 March 14-17, 2017 Vilnius, Lithuania.