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MODULE OF HOMEOMORPHISMS TO MODULE

Abstract

In this article, after a concise presentation of the modules over rings as a generalization of vector space over the fields, their homeomorphisms are treated. Further builds R-module si R-module of morphisms of the modules.

Keywords: R-module, left (right) R-module, abelian group, associative ring, R-homeomorphisms

1. The meaning of the R-Module, feature

Let ***M*** be an non empty set of equipped with an internal algebraic action [2] marked with the symbol of collection + and ***R*** an associative ring whatsoever [3]. A set ***M*** is also equipped with an algebraic external action [2] indicated by the multiplication symbol ·, which, when reflecting $\boldsymbol{R\times M}$ in ***M***, is referred to as the left multiplication in ***M*** with elements from ***R***, whereas, when reflecting the $\boldsymbol{M\times R}$ in ***M*** is called right multiplication in ***M*** with elements from ***R***. In the first case the couple's image $(r,m)\in R\times M$ is written $r{\cdot}m$, in the second case the couple's image $(m,r)\in M\times R$ is written $m\cdot r$.

Definition 1.1 [1, 5, 6] In the above conditions, the left module above the ***R*** ring is called the structure (***M***, +, ·), which has its own attributes:

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- (***M***, +) *is an abelian group*; (1)
- $\forall (r_1, r_2, m) \in R^2 \times M$, $r_1(r_2m) = (r_1r_2)m$; (2)
- $\forall (r, m_1, m_2) \in R \times M^2$, $r(m_1 + m_2) = rm_1 + rm_2$; (3)
- $\forall (r_1, r_2, m) \in R^2 \times M$, $(r_1 + r_2)m = r_1m + r_2m$. (4)

Definition 1.2. Under the above conditions, the right module above the ***R*** ring is called the structure (***M***, +, ·), which has its own attributes:

- (***M***, +) *is an abelian group*; (1')
- $\forall (m, r_1, r_2) \in M \times R^2$, $(mr_1)r_2 = m(r_1r_2)$; (2')
- $\forall (m_1, m_2, r) \in M^2 \times R$, $(m_1 + m_2)r = m_1r + m_2r$; (3')
- $\forall (m, r_1, r_2) \in M \times R^2$, $m(r_1 + r_2) = mr_1 + mr_2$. (4')

The left (right) module above the *R* ring is marked $_RM$ (M_R) and is called ***R***-left module (right). If the left-hand module above ***R*** is also the right is called a *module* above the ***R*** ring, in short *R*-module.

If the ring has a single element [1] 1_R (short 1) and the above-mentioned attributes for $_RM$ (M_R) is added the feature

- $\forall m \in M$, $1 \cdot m = m$ ($m \cdot 1 = m$) (5)

then the module $_RM$ (M_R) is called *the unitary left (right) module* above the *R* ring.

In ongoing, the ***R*** ring is associated and for a module on such a ring simple naming is used ***R****-Module*.

Below we will treat the ***R***-modules, implying left ***R***-modules, since the right ***R***-modules are treated analogously.

THEOREM 1.1. A ***R***-module ***M*** enjoys the following attributes:

- $\forall m \in M$, $0_R \cdot m = 0_M$; (6)
- • $\forall r \in R$, $r \cdot 0_M = 0_M$; (7)
- • • $\forall m \in M$, $\forall r \in R$, $(-r) \cdot m = r \cdot (-m) = -r \cdot m \in M$. (8)

Proof. Let r be a fixed element of the R ring and m any other element of the ${}_RM$ module. By Definition 1.1. we have $r \cdot m + 0_R \cdot m = (r + 0_R) \cdot m = r \cdot m$. On the other hand, by the additive group $(M, +)$, we have $r \cdot m + 0_M = r \cdot m$. From here $r \cdot m + 0_R \cdot m = r \cdot m + 0_M$, that gives $0_R \cdot m = 0_M$.

- • $r \cdot 0_M = r \cdot (0_R \cdot m) = (r \cdot 0_R) \cdot m = 0_R \cdot m = 0_M$.
- • • $r \cdot m + (-r) \cdot m = (r + (-r))m = 0_R \cdot m = 0_R \cdot m = 0_M$

$\Rightarrow (-r) \cdot m = -r \cdot m.$

2. *R*-Homeomorphisms of *R*-Modules

Definition 2.1 [1,6] *R-homomorphism (or R-morphism) of a R-module M in a R-module N is called any reflection f: M→N having attributes*

- üüüü+ ₂ = ₁ + ₂ $\forall m_1, m_2 \in M;$ (9)
- $f(r \cdot m) = r \cdot f(m), \ \forall r \in R$ and $\forall m \in M$ (10)

(ose $f(m \cdot r) = f(m) \cdot r, \ \forall r \in R$ and $\forall m \in M$).

If $M=N$, then the reflection f is called R-endomorphism in M.

THEOREM 2.1. For every two R-modules $M, N,$ if the reflection $f: M \to N$ is a R-homomorphism, then

- $f(0_M) = 0_N,$ (11)
- $f(-m) = -f(m), \forall m \in M$, (12)
- $f(m_1 - m_2) = f(m_1) - f(m_2), \forall m_1, m_2 \in M$, (13)

Proof. According to (6) and (10) we have

$f(0_M) = f(0_R \cdot m) = 0_N$ $f(\theta_M) = \theta_N$.

Further, according to (9),

$0_N = f(0_M) =$ ü(üü + (−)) = ( ) + (−) ,

that tells us $f(-m)$ is the symmetric of $f(m)$ in the group $(N, +)$, so $-f(m) = f(-m)$. Finally,

$$f(m_1 \text{ℵℵℵ} m_2 \text{ℵ} \quad f(m_1 \quad (\quad m_2)) \quad f(m_1) \quad f(\quad m_2)$$

$$= f(m_1) + (-f(m_2)) = f(m_1) - f(m_2), \forall m_1, m_2 \in M.$$

THEOREM 2.2. For each two ***R***-modules ***M, N,*** reflection $\boldsymbol{p_0}$: $\boldsymbol{M \to N}$, defined by $\boldsymbol{p_0(m)}=0_N$, $\forall m \in M$, is the ***R***-homeomorphism *of* ***M*** to ***N.***

Proof. From the above definition of reflection p_0 we have

$$p_0(m_1+m_2)=0_N=0_N+0_N=p_0(m_1)+p_0(m_2),\ \forall m_1,m_2\in M,$$
which indicates that p_0 enjoys the attribute(9); we also have
$$p_0(r\cdot m)=0_N=r\cdot 0_N=r\cdot p_0(m),\ \forall r\in R \text{ dhe } \forall m\in M,$$
which indicates that p_0 also enjoys the attribute (10).

THEOREM 2.3. Identical reflection $I_M : M \rightarrow M$ (e.g the reflection defined by $I_M(m)=m, \forall m\in M$ is an ***R***-endomorphism in ***M.***

Proof. From the above definition of the identical reflection I_M we have
$$I_M(m_1+m_2)=m_1+m_2=I_M(m_1)+I_M(m_2),\ \forall m_1,m_2\in M,$$
Indicating that the I_M enjoys the attribute (9); we also have

$$I_M(r\cdot m)=r\cdot m=r\cdot I_M(m),\ \forall r\in R \text{ dhe } \forall m\in M,$$
which indicates that I_M enjoys the attribute (10).

3. Module $Hom_R(M,N)$ of *R*-Homeomorphisms of the Modules

The study of homomorphismes of modules bring to the construction of an important module, called the *homomorphism module*.

Let be given the *R*-module *M* and the *R*-module *N*. The set of *R*-homomorphisms from *M* to *N* is written $Hom_R(M,N)$.

Definition 3.1. Let be f, g two possible reflections from *M* to *N* and r an element of an *R* ring. Then:

1. Many of the reflection *f* with the *g* reflection, which is written *f + g,* is called reflection $f+g$: $M \rightarrow N$, *defined* by
$$(f+g)(m)=f(m)+g(m)\ ,\forall m\in M\ . \quad (14)$$
2. The opposite reflection of f reflection, which is written *-f,* is called reflection $-f$: $M \rightarrow N$, defined by
$$(-f)(m)=-f(m)\ ,\forall m\in M\ . \quad (15)$$
3. The left product of the reflection *f* with the element $r\in R$,, which is written $r\cdot f$, is called the reflection $r\cdot f : M \rightarrow N$, defined by
$$(r\cdot f)(m)=r\cdot f(m)\ ,\forall m\in M\ . \quad (16)$$
An analogy is given to the meaning and the right production $f\cdot r$ such that
$(f\cdot r)(m)=f(m)\cdot r\ ,\forall m\in M$.

THEOREM 3.1. If the reflections f, g are ***R***-homomorphisms from ***M*** to ***N*** then:

1. $f+g\in Hom_R(M,N)$, (17)
otherwise, their amount $f+g$ is a ***R***-homomorphism from ***M*** to ***N;***

2. $-f\in Hom_R(M,N)$, (18)
otherwise, the reverse reflection $-f$ is a ***R***-homomorphism from ***M*** to ***N;***

3. For each r $\in R$, where R is commutative,

$$r \cdot f \in Hom_R(M,N), \qquad (19)$$

otherwise, the left (right) production of f reflection with elements from ***R*** is a ***R***-homomorphism from ***M*** to ***N.***

Proof.

1. Since the reflections f, g are R-homomorphisms from M to N, then

$$(f+g)(m_1+m_2) \overset{(14)}{=} f(m_1+m_2)+g(m_1+m_2)$$
$$\overset{(9)}{=} [f(m_1)+f(m_2)]+[g(m_1)+g(m_2)]$$
$$=[f(m_1)+g(m_1)]+[f(m_2)+g(m_2)]$$
$$\overset{(14)}{=} (f+g)(m_1)+(f+g)(m_2),\ \forall m_1,m_2 \in M,$$

which shows that f+g enjoys the attribute (9); we also have

$$(f+g)(rm) \overset{(14)}{=} f(rm)+g(rm)$$
$$\overset{(10)}{=} rf(m)+rg(m)$$
$$= r[f(m)+g(m)]$$
$$\overset{(14)}{=} r[(f+g)(m)],\ \forall r \in R \text{ dhe } \forall m \in M,$$

which shows that f+g enjoys the attribute (10). Consiquently f+g $\in$ $Hom_R(M,N)$

2. Reflection f is R-homomorphism from M to N, therefore

$$(-f)(m_1+m_2) \overset{(15)}{=} -f(m_1+m_2) \overset{(12)}{=} f(-(m_1+m_2)) = f(-(m_1)+(-m_2))$$
$$\overset{(9)}{=} f(-m_1)+f(-m_2) \overset{(12)}{=} (-f(m_1))+(-f(m_2)) \overset{(15)}{=} (-f)(m_1)+(-f)(m_2),\ \forall m_1,m_2 \in M,$$

which shows that $-f$ enjoys the attribute (9); Also, having in mind and (8) we have

$$(-f)(r \cdot m) \overset{(12)}{=} f(-r \cdot m) \overset{(8)}{=} f(r \cdot (-m)) \overset{(10)}{=} r \cdot f(-m) \overset{(12)}{=} r \cdot [-f(m)]$$
$$\overset{(15)}{=} r \cdot [(-f)(m)],\ \forall r \in R \text{ dhe } \forall m \in M,$$

which shows that $-f$ enjoys even the attribute (10).

3. We also have

$$(r \cdot f)(m_1+m_2) \overset{(16)}{=} r \cdot f(m_1+m_2) \overset{(9)}{=} r \cdot [f(m_1)+f(m_2)] = r \cdot f(m_1)+r \cdot f(m_2)$$
$$\overset{(16)}{=} (r \cdot f)(m_1)+(r \cdot f)(m_2),\ \forall m_1,m_2 \in M,$$

showing that $r{\cdot}f$ has its attribute (9); also, knowing that the R ring is commutative we have

$$(r{\cdot}f)(\rho m) \overset{(16)}{=} r{\cdot}f(\rho m)] \overset{(10)}{=} r{\cdot}[\rho f(m)] = (r\rho){\cdot}f(m) = (\rho r){\cdot}f(m)$$

$$= \rho\cdot[r{\cdot}f(m)] \overset{(16)}{=} \rho\cdot[(r{\cdot}f)(m)],\ \forall\rho\in R \text{ dhe } \forall m\in M,$$

which shows that $r{\cdot}f$ also enjoys attribute (10).

Definition 3.2. ***R***-homomorphism ***f+g: M→N*** is called ***R***-homeomorphism ***f: M→N*** with ***R***-homeomorphism ***g: M→N, R***-homeomorphism ***-f*** is called the opposite ***R***-homeomorphism ***f: M→N,*** but **R**-homomorphism r·f (f·r), when ***R*** is commutative, is called left (right) production of ***R***-homomorphism ***f: M→N*** with element r∈R

Through this definition, they are introduced into the set $Hom_R(M,N)$ action of addition + and left (right) multiplication, which make it algebra ($Hom_R(M,N)$, +, ·) with two actions.

THEOREM 3.2. If the ***R*** ring is commutative, then the algebra ($Hom_R(M,N)$, +, ·) of ***R***-homeomorphisms from ***M*** to ***N*** is the ***R***-left(right) module.

Proof. We show that they satisfy the conditions (1), (2), (3), (4) of Definition 1.1. of a left R-module.

(1) From the above it is easy to see that:

- $\forall f,g,h\in Hom_R(M,N)$, $(f+g)+h=f+(g+h)$;
- $\forall f\in Hom_R(M,N)$, $f+p_0=f$;
- $\forall f\in Hom_R(M,N)$, $f+(-f)=p_0$;
- $\forall f,g\in Hom_R(M,N)$, $f+g=g+f$,

indicating that $Hom_R(M,N)$, +) is an abelian group.

(2) $\forall(r_1,r_2,f)\in R^2\times Hom_R(M,N)$, writing $g=r_2\cdot f$, we have

$$[r_1\cdot(r_2{\cdot}f)](m) = (r_1\cdot g)(m) \overset{(16)}{=} r_1\cdot g(m) = r_1\cdot[(r_2{\cdot}f)(m)] \overset{(16)}{=} r_1\cdot[r_2{\cdot}f(m)]$$

$$\overset{(10)}{=} r_1\cdot[f(r_2\cdot m)] \overset{(10)}{=} f(r_1\cdot(r_2\cdot m)) = f((r_1\cdot r_2)\cdot m) \overset{(10)}{=} (r_1\cdot r_2){\cdot}f(m)$$

$$\overset{(16)}{=} [(r_1\cdot r_2){\cdot}f](m),\ \forall m\in M,$$

which indicates that $r_1\cdot(r_2{\cdot}f) = (r_1\cdot r_2){\cdot}f$.

(3) $\forall(r,f,g)\in R\times[Hom_R(M,N)]^2$ we have

$$[r{\cdot}(f+g)](m) \overset{(16)}{=} r{\cdot}[(f+g)(m)] \overset{(14)}{=} r{\cdot}[f(m)+g(m)] = r{\cdot}f(m)+r{\cdot}g(m)$$

$$\overset{(16)}{=} (r{\cdot}f)(m)+(r{\cdot}g)(m) \overset{(14)}{=} (r{\cdot}f+r{\cdot}g)(m),\ \forall m \in M,$$

which indicates that $r{\cdot}(f+g)= r{\cdot}f+r{\cdot}g$.

(4) $\forall (r_1, r_2, f) \in R^2 \times Hom_R(M, N)$, we have

$$[(r_1+r_2){\cdot}f](m) \overset{(16)}{=} (r_1+r_2){\cdot}f(m) \overset{(10)}{=} f((r_1+r_2)m)=f(r_1m+r_2m) \overset{(9)}{=} f(r_1m)+f(r_2m)$$

$$\overset{(10)}{=} r_1{\cdot}f(m)+r_2{\cdot}f(m) \overset{(16)}{=} (r_1{\cdot}f)(m)+(r_2{\cdot}f)(m) \overset{(9)}{=} (r_1{\cdot}f+r_2{\cdot}f)(m), \forall m \in M,$$

which indicates that $(r_1+r_2){\cdot}f= r_1{\cdot}f+r_2{\cdot}f$.

Analogously it is shown that ($Hom_R(M, N)$, +, ·) is the right R-module when · is right multiplication with elements from R.

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