Homomorphism in Weakly Γ-Ring

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Abstract

Many algebraic structures have been defined so far. One of them, is that of Γ-ring, which is a generalization of ring. Weakening some of the conditions of the definition of Γ-ring, it has also been defined the concept of weakly Γ-ring. An important and well-known concept for every algebraic structure is homomorphism. In this paper, the concept of homomorphism in weakly Γ-ring is introduced. Further, some simple results analogous to the theory of rings, related to this concept are extended.

Keywords: *Γ-semigroup, Γ-ring, weakly Γ-ring, homomorphism*

1. Introduction

The concept of Γ ring, which is a generalization of the concept of ring, was first defined by Nobusawa in [1].

Barnes, in [2], weakened some of the conditions of the definition of Nobusawa, and defined those that he called Γ-rings, naming Γ-rings defined in [1], as Γ-rings of Nobusawa.

Based to the definition of Nobusawa's Γ-ring, Sen in [3], defined Γ-semigroup that is called Γ-semigroup of Sen. Sen and Saha in [4], defined a generalization of Γ-semigroup of Sen, which is called a Γ-semigroup. The concept of Γ-semigroup may be obtained by that of Γ-ring, by extracting addition.

Petro and Sema, in [5], weakened further the conditions of the definition of Barnes and defined those that they called weakly Γ-rings.

An important concept for every algebraic structure is homomorphism. Thus, it is eligible extending this concept to weakly Γ-rings.

In this paper, homomrphism to weakly Γ-rings is introduced and some simple results of rings to Γ-rings, which are related to this concept, are extended.

2. Materials and Methods

Here we give some notions and present some auxiliary results that will be used throughout the paper.

Let M and Γ be two nonempty sets. Any map from $M \times \Gamma \times M$ to M is called a Γ-*multiplication* on M and is denoted by $(\cdot)_\Gamma$. The result of this Γ-multiplication for each $a, b \in M$ and each $\gamma \in \Gamma$, is denoted by $a\gamma b$.

The concept of Γ-ring, which is a generalization of the concept of ring, was first defined by Nobusawa in [1], as follows:

Definition 2.1. [1] *Let M be an additive group with elements $a, b, c, \ldots$ and Γ another additive group with elements $\alpha, \beta, \gamma, \ldots$. Assume that $a\alpha b$ is defined as an element of M and $\alpha a \beta$ is defined as an element of Γ for each a, b, α and β. If the products satisfy the following conditions*:

1. $(a_1+a_2)\alpha b=a_1\alpha b+a_2\alpha b$, $a(\alpha_1+\alpha_2)b=a\alpha_1 b+a\alpha_2 b$, $a\alpha(b_1+b_2)=a\alpha b_1+a\alpha b_2$,
2. $(a\alpha b)\beta c = a\alpha(b\beta c) = a(\alpha b\beta)c$,
3. *if $a\alpha b = 0$ for each a and b in M, then $\alpha = 0$,*

then M is called a Γ-ring.

An ordinary ring $(A, +, \cdot)$ may turn into a Γ-ring, if we get M and Γ to be equal to A.

Barnes, in [2], weakened some of the conditions of Nobusawa, by calling Γ-rings of Nobusawa the ones defined as above and simply Γ-rings those that he defined himself.

Definition 2.2. [2] Every ordered five-tuple $(M, \Gamma, +, \oplus, (\cdot)_\Gamma)$, where M, Γ are sets, + is an addition on M, $\oplus$ addition on Γ, $(\cdot)_\Gamma$ is a Γ-multiplication on M, such that:

1. *$(M, +)$ is an abelian group.*
2. *$(\Gamma, \oplus)$ is an abelian group.*
3. $\forall(a, b, c, \alpha, \beta) \in M^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c)$.
4. $\forall(a, b, c, \gamma) \in M^3 \times \Gamma, (a + b)\gamma c = a\gamma c + b\gamma c$.
5. $\forall(a, b, c, \gamma) \in M^3 \times \Gamma, a\gamma(b + c) = a\gamma b + a\gamma c$.
6. $\forall(a, b, \alpha, \beta) \in M^2 \times \Gamma^2, a(\alpha \oplus \beta)b = a\alpha b + a\beta b$,

is called Γ-ring (of Barnes).

Sen and Saha in [4], defined Γ-semigroups, which may be obtained by the definition of Γ-rings, by avoiding the additions:

Definition 2.3. [4] Every ordered pair $(M, (\cdot)_\Gamma)$, where M and Γ are two nonempty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on M, which satisfies the following condition:

$\forall(a, b, c, \alpha, \beta) \in M^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c)$,

is called Γ-semigroup.

The element $0 \in M$, such that:

$\forall(a, \gamma) \in M \times \Gamma, a\gamma 0 = 0 = 0\gamma a$,

is called zero of the Γ-semigroup $(M, (\cdot)_\Gamma)$.

Petro and Sema in [5], weakened further the conditions of the definition of Γ-rings (of Barnes), by defining weakly Γ-rings, as follows:

Definition 2.4. [5] Every ordered triple $(M, +, (\cdot)_\Gamma)$, where M, Γ are two nonempty sets, $+$ is an addition on M and $(\cdot)_\Gamma$ is a Γ-multiplication on M, such that:

1) *$(M, +)$ is an abelian group*,
2) *$(M, (\cdot)_\Gamma)$ is a Γ-semigroup*,
3) $\forall(a, b, c, \gamma) \in M^3 \times \Gamma, [(a + b)\gamma c = a\gamma c + b\gamma c] \wedge [a\gamma(b + c) = a\gamma b + a\gamma c]$,

is called weakly Γ-ring.

We notice that plain rings, Γ-rings of Nobusawa and Γ-rings of Barnes, are weakly Γ-rings, but the converse is not true.

Saha and Seth in [6] have introduced the concept of homomrphism between two Γ-semigroups, as follows:

Definition 2.5. [6] Let $(M, (\cdot)_\Gamma)$ be a Γ-semigroup and $(M_1, (\cdot)_{\Gamma_1})$ a Γ_1-semigroup. A pair of mappings (h_1, h_2), where $h_1: M \to M_1$ and $h_2: \Gamma \to \Gamma_1$, such that

$h_1(a\gamma b) = h_1(a)h_2(\gamma)h_1(b)$,

for each $a, b \in M$ and $\gamma \in \Gamma$, is called a homomorphism of (M, Γ) to (M_1, Γ_1).

Let $(M, +, (\cdot)_\Gamma)$ be a weakly Γ-ring. Every nonempty subset T of M, such that $(T, +)$ is a subgroup of $(M, +)$ and $a\gamma b \in T$, for each $(a, b) \in T^2$ and $\gamma \in \Gamma$, is called subΓ-ring of M.

Let M be a weakly Γ-ring and A, B two nonempty subsets of M. Define:

$A\Gamma B = \{\sum_{i=1}^{n} a_i\gamma_i b_i \in M : a_i \in A, b_i \in B, \gamma_i \in \Gamma \text{ for each } i = 1,2, \ldots, n; n \in \mathbb{N}\}$.

Every subgroup R $[L]$ of the group $(M, +)$, such that:

$R\Gamma M \subseteq R$ $[M\Gamma L \subseteq L]$,

is called right [left] ideal of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$.

Every subgroup I of the group $(M, +)$, such that:

$I\Gamma M \subseteq I$ and $M\Gamma I \subseteq I$, is called *ideal* of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$.

Thus, I is an ideal of the weakly Γ-ring M, only if it is a left ideal and a right ideal of M simultaneously.

3. Conclusions

In this section, basing on what is given above, mixing them, some new results are given.

Definition 3.1. *Let $(M, +, (\cdot)_\Gamma)$ be a weakly Γ-ring and $(M', +, (\cdot)_{\Gamma'})$ be a weakly Γ'-ring. Every ordered pair of mappings $H = (h_1, h_2)$, where $h_1: M \to M'$ and $h_2: \Gamma \to \Gamma'$, such that*:

1) $\forall (a, b) \in M^2, h_1(a + b) = h_1(a) + h_1(b)$.
2) $\forall (a, \alpha, b) \in M \times \Gamma \times M, h_1(a\alpha b) = h_1(a)h_2(\alpha)h_1(b)$,

is called a homomorphism of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the weakly Γ'-ring $(M', +, (\cdot)_{\Gamma'})$.

It is obvious that every homomorphism of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the weakly Γ′-ring $(M', +, (\cdot)_{\Gamma'})$, is an ordered pair of mappings (h_1, h_2), where h_1 is a homomorphism of the additive group $(M, +)$ of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the additive group $(M', +)$ of the weakly Γ′-ring $(M', +, (\cdot)_{\Gamma'})$, whereas (h_1, h_2) is a homomorphism of the Γ-semigroup $(M, (\cdot)_\Gamma)$ of the Γ-multiplication of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the Γ′-semigroup $(M', (\cdot)_{\Gamma'})$ of the Γ′-multiplication of the weakly Γ′-ring $(M', +, (\cdot)_{\Gamma'})$.

If both mappings h_1, h_2 are injective (one-to-one), the homomorphism H=(h_1, h_2) is called monomorphism and (M, Γ) is called monomorph to (M', Γ'). If these mappings are both surjective (onto), H is called epiomorphism and (M, Γ) is called epiomorph to (M', Γ'), whereas when both h_1, h_2 are bijective, H is called isomorphism, and (M, Γ) is called isomorph to (M', Γ').

Definition 3.2. Let $H = (h_1, h_2)$ be a homomorphism of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the weakly Γ'-ring $(M', +, (\cdot)_{\Gamma'})$. The kernel of the homomorphism h_1 of the additive group $(M, +)$ to the additive group $(M', +)$, is called kernel of the homomorphism $H = (h_1, h_2)$, and will be denoted by $KerH$. Thus:

$KerH = \{x \in M: h(x) = 0_{M'}\}$.

The image of the homomorphism h_1, is called *image* of the homomorphism $H = (h_1, h_2)$, and will be denoted by ImH. Thus:

$ImH = \{h_1(x) \in M': x \in M\} = \{x' \in M': \exists x \in M, h_1(x) = x'\}$.

Proposition 3.3. *For every homomorphism $H = (h_1, h_2)$ of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the weakly Γ'-ring $(M', +, (\cdot)_{\Gamma'})$, the kernel $KerH$ is an ideal of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$, whereas, when h_2 is a surjective mapping, ImH is a subΓ'-ring of the weakly Γ'-ring $(M', +, (\cdot)_{\Gamma'})$.*

Proof. $KerH$ is a subgroup of the additive group of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$. For each element a of M, for each element b of $KerH$ and for each element γ of Γ, the following hold

$h_1(a\gamma b) = h_1(a)h_2(\gamma)h_1(b) = 0h_2(\gamma)h_1(b) = 0,$

$h_1(b\gamma a) = h_1(b)h_2(\gamma)h_1(a) = h_1(b)h_2(\gamma)0 = 0,$

which show that $KerH$ is an ideal of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$.

Let a', b' be two elements of ImH and γ' an arbitrary element of Γ′. Then, there exist the elements $a, b \in M$ and the element $\gamma \in \Gamma$, such that:

$a' = h_1(a), \;\; b' = h_1(b), \;\; \gamma' = h_2(\gamma)$.

The following equalities hold:

$a'\gamma' b' = h_1(a)h_2(\gamma)h_1(b) = h_1(a\gamma b),$

which show that ImH is a subΓ′-ring of the weakly Γ′-ring $(M', +, (\cdot)_{\Gamma'})$, since ImH is a subgroup of the group $(M', +)$. ■

Let $H = (h_1, h_2)$ be a homomorphism of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the weakly Γ′-ring $(M', +, (\cdot)_{\Gamma'})$, and B′ a subset of M'. Denote

$H^{-1}(B') = \{x \in M: h_1(x) \in B'\}$.

The subset $H^{-1}(B')$ of M, will be called *inverse image* of B′.

Proposition 3.4 *Let $H = (h_1, h_2)$ be an epiomorphism of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ to the weakly Γ'-ring $(M', +, (\cdot)_{\Gamma'})$ with kernel, the ideal I of $(M, +, (\cdot)_\Gamma)$. Then, a nonempty subset B' of M' is an ideal of $(M', +, (\cdot)_{\Gamma'})$ if and only if*

$H^{-1}(B') = B$

is an ideal of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$ that contains the ideal I.

Proof. Assume that the nonempty subset B′ of M' is an ideal of the weakly Γ′-ring $(M', +, (\cdot)_{\Gamma'})$. Let a be an ordinary element of M, b an ordinary element of $B = H^{-1}(\mathrm{B}')$ and γ an ordinary element of Γ. Since:

$h_1(a\gamma b) = h_1(a)h_2(\gamma)h_1(b) \in M'\Gamma'B' \subseteq B'$,

$h_1(b\gamma a) = h_1(b)h_2(\gamma)h_1(a) \in B'\Gamma'M' \subseteq B'$,

the elements $a\gamma b, b\gamma a$ belong to the subset B and consequently $H^{-1}(\mathrm{B}') = B$ is an ideal of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$, since $H^{-1}(\mathrm{B}')$ is a subgroup of the additive group $(M, +)$ of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$.

The ideal $H^{-1}(\mathrm{B}') = B$ contains the ideal I of the epiomorphism H,:since we have

$\forall x \in M, x \in I \Rightarrow h_1(x) = 0 \in B'$.

Conversely, suppose that $H^{-1}(\mathrm{B}') = B$ is an ideal of the weakly Γ-ring $(M, +, (\cdot)_\Gamma)$, that contains the ideal I. Let a' be an element of M', b' an element of B' and γ' an element of Γ'. There exist the elements $a \in M, b \in H^{-1}(\mathrm{B}') = B$ and $\gamma \in \Gamma$, such that:

$a' = h_1(a), \;\; b' = h_1(b), \;\; \gamma' = h_2(\gamma)$.

The following equalities hold:

$a' - b' = h_1(a) - h_1(b) = h_1(a - b)$,

$a'\gamma'b' = h_1(a)h_2(\gamma)h_1(b) = h_1(a\gamma b)$,

$b'\gamma'a' = h_1(b)h_2(\gamma)h_1(a) = h_1(b\gamma a)$,

that show that B' is an ideal of the weakly Γ'-ring $(M', +, (\cdot)_{\Gamma'})$, since $B' \neq \emptyset$, because $I \subseteq H^{-1}(B')=B$, and $H^{-1}(\mathrm{B}') = B$ is an ideal of $(M, +, (\cdot)_\Gamma)$, that guarantees us that a-b, aγb, bγa are elements of $H^{-1}(\mathrm{B}') = B$. ■

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